

# Local Approximability of Minimum Dominating Set on Planar Graphs

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**Abstract.** We show that there is no deterministic local algorithm (constant-time distributed graph algorithm) that finds a  $(7 - \epsilon)$ -approximation of a minimum dominating set on planar graphs, for any positive constant  $\epsilon$ . In prior work, the best lower bound on the approximation ratio has been  $5 - \epsilon$ ; there is also an upper bound of 52.

## 1 Introduction

This work studies one of the last uncharted corners in the area of deterministic local algorithms: planar graphs.

A *local algorithm* is a distributed graph algorithm that runs in  $O(1)$  communication rounds, independently of the size of the network. While the theory of *randomised* local algorithms is still in its infancy, we have nowadays a good understanding of the capabilities of *deterministic* local algorithms.

For many classical graph problems, there are exactly matching upper and lower bounds on the best possible approximation ratio that can be achieved by a deterministic local algorithm [6]. In many cases, we can apply a straightforward two-step procedure to derive tight lower bounds:

1. Prove tight bounds for anonymous networks (without unique identifiers).
2. Apply a simulation argument [2] to show that unique identifiers do not help.

However, there are some isolated examples of natural questions in which the above two-step procedure fails badly. Perhaps the most intriguing example is *dominating sets on planar graphs*:

1. We do not have tight bounds for this problem in anonymous networks.
2. Planar graphs are not closed under lifts, and therefore the simulation argument [2] cannot be applied.

In this work we are interested in the smallest  $\alpha$  such that there is a deterministic local algorithm that finds an  $\alpha$ -approximation of a minimum dominating set in any planar graph. The current bounds are very far from being tight:

- $5 - \epsilon < \alpha \leq 636$  for anonymous networks [1, 7],
- $5 - \epsilon < \alpha \leq 52$  in the LOCAL model [1, 3, 4, 8].

In this work we give the first improvement on the lower bounds in six years: we prove a lower bound  $\alpha > 7 - \epsilon$  for both models, for any positive constant  $\epsilon$ .

## 2 Proof Overview

Let  $\mathcal{A}$  be a deterministic distributed algorithm with running time  $T = O(1)$  in the LOCAL model. Assume that  $\mathcal{A}$  finds a dominating set  $D = \mathcal{A}(G)$  in any planar graph  $G$ .

Pick sufficiently large  $m \gg T$  and  $r$ . Let  $m' = m - 2T$ . We will construct a planar graph  $G$  with  $n = m^2 r$  nodes as shown in Figure 1a. There are  $r$  *blocks* with  $m \times m$  nodes in each

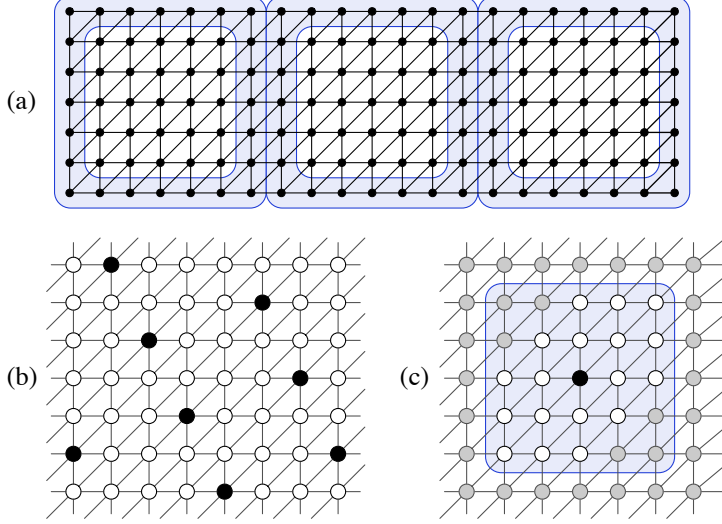


Figure 1: (a) Construction of graph  $G$  for  $T = 1$ ,  $m = 7$ , and  $r = 3$ . There are 3 blocks. In each block there are  $7 \times 7$  nodes:  $5 \times 5$  internal nodes (white area), surrounded by a boundary area of width 1 (shaded). (b) A dominating set  $D^*$  of  $G$  that contains only a fraction  $1/7$  of internal nodes. (c) The local output of an internal node  $v$  (black node) only depends on its radius- $T$  neighbourhood (white nodes, here  $T = 2$ ). In particular, if we know the unique identifiers in the  $k \times k$  region  $R_v$  around  $v$  (shaded area), we know the local output of node  $v$ .

block. The nodes of each block are partitioned to *internal nodes* and *boundary nodes*: there are  $m' \times m'$  internal nodes, and they are surrounded by boundary areas of width  $T$ . Let  $B_i$  be the set of nodes in block  $i$ , and let  $I_i \subseteq B_i$  be the set of internal nodes in block  $B_i$ . We will prove the following lemma.

**Lemma 1.** *For any  $m$  and any sufficiently large  $r$ , we can assign unique identifiers in  $G$  so that  $I_i \subseteq \mathcal{A}(G)$  for all  $1, 2, \dots, r - \ell$ , for some  $\ell = o(r)$ .*

In other words, all internal nodes of blocks  $1, 2, \dots, r - \ell$  are in the dominating set  $D = \mathcal{A}(G)$  produced by algorithm  $\mathcal{A}$ . Now if we choose large enough  $m$  and  $r$ , we can make the contributions of the boundary nodes and the contributions of the remaining  $o(r)$  blocks arbitrarily small. In particular, for any positive constant  $\epsilon'$ , we can pick  $m$  and  $r$  such that  $|D| \geq (1 - \epsilon')n$ .

On the other hand, there is a dominating set  $D^*$  which contains only a fraction  $1/7$  of the internal nodes; see Figure 1b. Therefore  $|D^*| \leq (1/7 + \epsilon')n$ , and the claim follows: for any positive constant  $\epsilon$  we can show that algorithm  $\mathcal{A}$  cannot find a factor  $7 - \epsilon$  approximation of a minimum dominating set on planar graphs.

### 3 Proof of Lemma 1

The proof uses the strategy of repeated applications of Ramsey's theorem; cf. Czygrinow et al. [1, Lemma 4]. We will use the notation  $\mathcal{A}(G, v) \in \{0, 1\}$  to refer the *local output* of node  $v$  when we apply algorithm  $\mathcal{A}$  to graph  $G$ ; we have  $\mathcal{A}(G, v) = 1$  if node  $v$  is in the dominating set computed by algorithm  $\mathcal{A}$ . By definition,  $\mathcal{A}(G, v)$  only depends on the radius- $T$  neighbourhood of  $v$  in  $G$ .

Let  $k = 2T + 1$ ,  $K = k^2$ , and  $M = m^2$ . Consider any internal node  $v \in I_i$  of any block  $B_i$ . The structure of graph  $G$  in the radius- $T$  neighbourhood does not depend on the choice of  $v$ . Hence the local output of node  $v$  only depends on the unique identifiers in the local neighbourhood. The local neighbourhood is contained within a rectangular  $k \times k$  region  $R_v \subseteq B_i$ ; see Figure 1c.

Let  $V = \{1, 2, \dots, n\}$  be the set of unique identifiers. Consider any  $K$ -subset of identifiers  $X \subseteq V$ ,  $|X| = K$ . We will associate a *colour*  $c(X) \in \{0, 1\}$  with each such set, as follows:

1. Pick an internal node  $v$ .
2. Assign the identifiers from  $X$  to region  $R_v$  in an increasing order by rows: the smallest  $k$  identifiers to the bottom row from left to right, etc. Assign the identifiers from  $V \setminus X$  to the remaining nodes arbitrarily.
3. Apply algorithm  $\mathcal{A}$ , and set  $c(X) = \mathcal{A}(G, v)$ .

Now we have defined a colouring of all  $K$ -subsets of  $V$ ; by restriction, we also have a colouring of all  $K$ -subsets of any  $V' \subseteq V$ . We say that  $Y \subseteq V$  is *monochromatic* if  $c(X_1) = c(X_2)$  for any  $K$ -subsets  $X_1$  and  $X_2$  of  $Y$ . By Ramsey's theorem [5] there exists an integer  $N = N(K, M)$  such that the following holds: if  $V'$  is any  $N$ -subset of  $V$ , then there always exists a monochromatic subset  $Y \subseteq V'$  of size  $M$ .

Now we will pick  $r$  and  $\ell$  so that  $\ell M > N$  and  $\ell = o(r)$ . Let  $V_1 = V$ . For each  $i = 1, 2, \dots, r - \ell$ , we define the identifiers of block  $i$  as follows.

1. As  $|V_i| \geq N$ , we can find a monochromatic subset  $Y_i \subseteq V_i$  of size  $M$ .
2. Assign the identifiers from  $Y_i$  to block  $B_i$  in an increasing order by rows: the smallest  $m$  identifiers to the bottom row from left to right, etc.
3. Set  $V_{i+1} = V_i \setminus Y_i$ .

Finally, assign the remaining  $\ell M$  identifiers from  $V_{r-\ell+1}$  to blocks  $r - \ell + 1, \dots, r$  arbitrarily.

To complete the proof, consider a block  $i$ , where  $1 \leq i \leq r - \ell$ . Let  $v \in I_i$  be an internal node of the block. Consider the  $k \times k$  region  $R_v$  around  $v$ , and let  $X_v$  be the set of unique identifiers assigned to region  $R_v$ . Observe that the identifiers of  $X_v$  are assigned in an increasing order by rows. It follows that  $\mathcal{A}(G, v) = c(X_v)$ , i.e., the local output of the internal node  $v$  is simply the colour of subset  $X_v$ . Furthermore,  $X_v \subseteq Y_i$  and  $Y_i$  was monochromatic. Hence all internal nodes of block  $i$  produce the same output. The common output cannot be 0; otherwise there would be nodes that are not dominated. Hence  $I_i \subseteq \mathcal{A}(G)$ .

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## References

- [1] A. Czygrinow, M. Hańćkowiak, and W. Wawrzyniak. Fast distributed approximations in planar graphs. In *Proc. 22nd International Symposium on Distributed Computing (DISC 2008)*, volume 5218 of *Lecture Notes in Computer Science*, pages 78–92. Springer, 2008. doi:10.1007/978-3-540-87779-0\_6.
- [2] M. Göös, J. Hirvonen, and J. Suomela. Lower bounds for local approximation. *Journal of the ACM*, 60(5):39:1–23, 2013. doi:10.1145/2528405. arXiv:1201.6675.
- [3] C. Lenzen. *Synchronization and Symmetry Breaking in Distributed Systems*. PhD thesis, ETH Zurich, January 2011.
- [4] C. Lenzen, Y. A. Pignolet, and R. Wattenhofer. Distributed minimum dominating set approximations in restricted families of graphs. *Distributed Computing*, 26(2):119–137, 2013. doi:10.1007/s00446-013-0186-z.
- [5] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30:264–286, 1930. doi:10.1112/plms/s2-30.1.264.
- [6] J. Suomela. Survey of local algorithms. *ACM Computing Surveys*, 45(2):24:1–40, 2013. doi:10.1145/2431211.2431223. http://www.cs.helsinki.fi/local-survey/.
- [7] W. Wawrzyniak. Brief announcement: a local approximation algorithm for MDS problem in anonymous planar networks. In *Proc. 32nd Annual ACM Symposium on Principles of Distributed Computing (PODC 2013)*, pages 406–408. ACM Press, 2013. doi:10.1145/2484239.2484281.
- [8] W. Wawrzyniak. A strengthened analysis of a local algorithm for the minimum dominating set problem in planar graphs. *Information Processing Letters*, 114(3):94–98, 2014. doi:10.1016/j.ipl.2013.11.008.